Basic methods for solving functional equations

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ABSTRACT
This article deals with functional equations, methods of solving them. At the end of the article, summing up the results, a general method for solving some functional equations is presented. And also, in conclusion, some examples of groups of functions are given that can be used to solve functional equations.

Keywords
Functional equation, functional Cauchy equation, continuity of a function, auxiliary function, substitution method, concept of a function group.
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Introduction

It is generally recognized that solving problems is the most important means of forming a system of basic mathematical knowledge, abilities and skills in students, the leading form educational activity of students in the process of learning mathematics, is one of the main means of their mathematical development.

By orienting students to the search for beautiful, elegant solutions to mathematical problems, the teacher thereby contributes to the aesthetic education of students and the improvement of their mathematical culture. And yet, the main goal of the tasks is to develop the creative and mathematical thinking of students, to interest them in mathematics, to lead to the discovery of mathematical facts.

The simplest examples of functional equations are: $f(x)=f(-x)$ is the parity equation, $f(x+T)=f(x)$ is the periodicity equation, etc.

A functional equation is an equation in which a function acts as an unknown. In this case, the solution to the equation is any function when substituted into the equation, it turns into an identity. To solve a functional equation means to find the set of all its solutions. For example, differential equations are special cases of functional equations.

Substitution method

This method consists in the fact that by applying instead of X (or Y) various substitutions and combining the resulting equations with the original, we obtain (usually by elimination) an algebraic equation for the desired function.

Example 1. Solve the equation

$$f\left(x+\frac{1}{x}\right)=\left(x^2+\frac{1}{x^2}\right) \quad (x\neq 0)$$

Solution. Putting $x+\frac{1}{x}=t,$ we square this equality:

$$x^2+\frac{1}{x^2}+2=t^2 \rightarrow x^2+\frac{1}{x^2}=t^2-2$$

After passing to the new variable, the functional equation will take the form $f(t)=t^2-2$. Therefore, we found a function, this is $f(x)=x^2-2$.

However, it is necessary to check the solution found. Verification is needed, in particular, for following reason: the function $y = x + \frac{1}{x}$ has the property that $|y| = \left|x + \frac{1}{x}\right| \geq 2$, so the question remains does the found function $f(x) = x^2 - 2$ satisfy the functional equation for $|f(x)| < 2$.

We put the function in the original equation and check whether it really satisfies it for all $x \neq 0$. The obtained equality obviously holds for all real $x \neq 0$, therefore the function $f(x) = x^2 - 2$ will be the only solution of the functional equation.

Example 2. Solve the equation

$$f\left(\frac{x+1}{x+2}\right) + 2f\left(\frac{x-2}{x+1}\right) = x$$

Solution.

1) Let $\frac{x-2}{x+1} = t,$ then $x = \frac{t+2}{1-t}$ $(t \neq 0, t \neq 1)$

2) Substitute in the original equation we get

$$f\left(\frac{1}{t}\right) + 2f(t) = \frac{t+2}{1-t}$$

3) Replace t with $\frac{1}{t}$, we get

$$f(t) + 2f\left(\frac{1}{t}\right) = \left(\frac{\frac{1}{t}+2}{1-\frac{1}{t}}\right) = \frac{1+2t}{t-1}$$

4) So, we get two equations:

$$f\left(\frac{1}{t}\right) + 2f(t) = \frac{t+2}{t-1}$$

$$f(t) + 2f\left(\frac{1}{t}\right) = \frac{1+2t}{t-1}$$

5) Multiply both sides of the 1st equation *lg*(-2) and add them with the 2nd equation, we get

$$-2f\left(\frac{1}{t}\right) - 4f(t) = \frac{-2t-4}{1-t}$$

$$2f\left(\frac{1}{t}\right) + f(t) = \frac{1+2t}{1-t}$$

$$-3f(t) = \frac{2t+4+1+2t}{t-1} = \frac{4t+1}{t-1}$$

$$f(t) = \frac{4t+5}{3-3t}$$

Then

$$f(x) = \frac{4x+5}{3-3x}$$

Method of mathematical induction

Using this method, knowing $f(1)$, we find for integer n. T hen we find $f\left(\frac{1}{n}\right)$ and $f(r)$ for rational r. This approach is used in situations where functions. Where functions are defined on Q, and is very useful, especially in simple tasks.

Example 3. Find a function $f(x)$ defined on the set of natural numbers such that $f(1)=1$ and

$f(x+y)=f(x)+f(y)+xy$ for any natural numbers *x* and *y*.

Solution. Putting *y*=1, we get $f(x+1)=f(x)+x+1$

Hence, $f(x)=f(x-1)+x$

$$f(1)=1$$
$$f(2)=f(1)+2$$
$$f(3)=f(2)+3$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f(x-1)=f(x-2)+x-1$$
$$f(x)=f(x-1)+x$$

Adding all these equalities we get

$$f(x)=1+2+3+\cdots+x=\frac{x(x+1)}{2}=\frac{x^2+x}{2}$$

Example 4. Find all functions *f* such that for any numbers *x*, *y* equality

$$f(x)\cdot f(y)-xy=f(x)+f(y)-1.$$

Solution. Substituting *y*=1, after simple transformations we obtain that

$$f(x)=1+\frac{x}{f(1)-1}$$

Substituting *x*=*y*=1, into the original equality, we get that $f(1)=2$ or $f(1)=0$. Answer $f(x)=1+x$ or $f(x)=1-x$.

Derivative and functional equations

Example 5. Find all functions *f* :*R*→*R*, which for all *x*, *y*∈ *R* satisfy the equation

$$f(x)+f(y)=f(x)+f(y)$$

Solution. We put in this equality *y*=0. We get

$$f(x)=f(x)+f(0)$$

for any *x*∈*R*, hence $f(0)=0$. Next, we differentiate the equality with respect to the variable *x*, setting *y*=const. We have $f'(x+y)=f'(x)$ for any *y*∈ *R*. Hence we conclude that $f'(x)=K=const$, and $f(x)=Kx$

Example 6. Find all real differentiable functions *f*(*x*) satisfying the equation

$$f(x)=\frac{f(x)+f(y)}{1-f(x)\cdot f(y)}$$

Solution. Let x=*y*=0. We have

$$f(0)=\frac{2f(0)}{1-f^2(0)}$$

Whence *f*(0)=0. After transformations, we have

$$\frac{f(x+h)-f(x)}{h}=\frac{f(h)}{h}\cdot\frac{1+f^2(x)}{1-f(x)\cdot f(h)}$$

Passing to the limit at *h*→0, taking into account that

$$\lim_{h\to 0}f(h)=0,$$

we obtain

$$f'(x)=C\big(1+f^2(x)\big),$$

where $C=f'(0)$ we integrate

$$\int\frac{dx}{1+f^2}=\int Cdx$$

from where $arctgf(x) = Cx + C_1$ and $f(x) = tg(Cx + C_1)$. Since $f(0)=0$, then $C_1=0$ and $f(x) = tg(Cx)$ checking we make sure that all these functions are solutions of the original equation.

The following functional functions are called the Cauchy equation

$$1)\ f(x+y) = f(x) + f(y) \to f(x) = kx \quad (1)$$
$$2)\ f(x+y) = f(x)\cdot f(y) \to f(x) = a^x \quad (2)$$
$$3)\ f(xy) = f(x) + f(y) \to f(x) = \log_a x \quad (3)$$
$$4)\ f(xy) = f(x)\cdot f(y) \to f(x) = x^a \quad (4)$$

Method of reducing a functional equation by changing a variable and a function

Example 7. Find all continuous functions satisfying the equation

$$f(x+y) = f(x) + f(y) + 2xy$$

Solution. As on auxiliary function, we take the function

$$g(x) = f(x) - x^2$$

Substituting into the original equation $f(x) = g(x) + x^2$, we obtain

$$g(x+y) + (x+y)^2 = g(x) + x^2 + g(y) + 2xy,$$
$$g(x+y) = g(x) + g(y),$$

That is, the function $g(x)$ satisfies the first Cauchy equation, whence

$$f(x) = kx + x^2$$

Example 8. Find all continuous functions $f(0,\infty) \to R$, satisfying the equation

$$f(xy) = xf(y) + yf(x).$$

Solution. Divide the equation by xy, we get

$$\frac{f(xy)}{xy} = \frac{f(y)}{y} + \frac{f(x)}{x}$$

We introduce an auxiliary function $g(x) = \frac{f(x)}{x}$, then we obtain the equation

$$g(xy) = g(x) + g(y)$$

That is, the function $g(x)$ satisfies the third Cauchy equation, whence $g(x) = \log_a x$, or $f(x) = x\log_a x$

Method to the limit

Example 9. The function $f\colon R \to R$ is continuous at the point 0 and for any $x \in R$ the equivalent of

$$2f(2x) = f(x) + x$$

is fulfilled. Find all such $f(x)$.

Solution. Let the function $f(x)$ satisfy the condition of the problem. Then

$$f(x) = \frac{1}{2}f\left(\frac{x}{2}\right) + \frac{x}{4} = \frac{1}{2}\left(\frac{1}{2}f\left(\left(\frac{x}{4}\right) + \frac{x}{8}\right)\right) + \frac{x}{4} = \frac{1}{2}f\left(\frac{x}{4}\right) + \frac{x}{16} + \frac{x}{4} = \cdots$$

$$\ldots = \frac{1}{2^n}f\left(\frac{1}{2^n}\right) + \frac{x}{2^{2n}} + \frac{x}{2^{2n-2}} + \cdots + \frac{x}{4} = \lim_{n\to\infty}\frac{1}{2^n}f\left(\frac{1}{2^n}\right) + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{x}{4^k} = \frac{x}{3}$$

Verification shows that $f(x) = \frac{x}{3}$ is indeed a solution.

Example 10. Find a function $f(x)$, bounded on any finite interval and satisfying the equation

$$f(x) = \frac{1}{2}f\left(\frac{x}{2}\right) = x - x^2$$

Solution. $x = 0 \to f(x) = 0$

$$\frac{1}{2}f\left(\frac{x}{2}\right)-\frac{1}{4}f\left(\frac{x}{4}\right)=\frac{x}{4}-\frac{x^2}{8},$$

$$\frac{1}{4}f\left(\frac{x}{4}\right)-\frac{1}{8}f\left(\frac{x}{8}\right)=\frac{x}{16}-\frac{x^2}{64},$$

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$$\frac{1}{2^n}f\left(\frac{x}{2^n}\right)-\frac{1}{2^{n+1}}f\left(\frac{x}{2^{n+1}}\right)=\frac{x}{4^n}-\frac{x^2}{8^n}$$

Let's add all these equalities

$$f(x)-\frac{1}{2^{n+1}}f\left(\frac{x}{2^{n+1}}\right)=x\left(1+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^n}\right)-x^2\left(1+\frac{1}{8}+\frac{1}{8^2}+\cdots+\frac{1}{8^n}\right)$$

Let us pass to the limit at $n \to \infty$. Taking into account the boundedness of $f(x)$, at zero and that $f(0) = 0$, we obtain

$$f(x)=\frac{4}{3}x-\frac{8}{7}x^2$$

Search substitutions

Example 11. Find all the functions of $f: R \to R$ that for all $x, y \in R$ satisfy the equation

$$f(x+y^2+2y+1)=y^4+4y^3+2xy^2+5y^2+4xy+2y+x^2+x+1 \quad (*)$$

Solution. Since we want to get the expression $f(x)$, let's try to get rid of the term $y^2 + 2y + 1$ under the function sign. The equation $y^2 + 2y + 1 = 0$ has one solution $y = -1$. Substituting $y = -1$ in (*), we get

$$f(x)=x^2-x+1.$$

Example 12. Find all functions of $f: R \to R$ which for all $x, y \in R$ satisfy the equation

$$f\big((x^2+6x+6)y\big)=y^2x^4+12y^2x^3+48y^2x^2-4yx^2+$$
$$+72y^2x-24yx+36y^2-24y \qquad (**)$$

Solution. As in the previous example, we want to get a free variable $(X \text{ or } Y)$ under the function sign. In this case, it is obviously easier to get Y. Solving the equation $(x^2 + 6x + 6)y = y$ for x, we get $x_1 = -1$, $x_2 = -5$. Substituting any of these values into (**) gives us $f(y) = y^2 - 4y$.

A method for solving some functional equations using the concept of a function group.

Definition. An arbitrary set G of a function defined on some set M is called a group with respect to operation °, if it has the same properties as the system (g_1,g_2,g_3, g_4), that is

1. For any two functions $f \in G, g \in G$, their composition $f^\circ g$ also belongs ° G .
2. The function e(x)=x belongs G.
3. For any function $f \in G$, there exists an inverse functions f^{-1}, which also belongs to G .

This definition is a special case of the general definition of the concept of a group, one of the most important concepts of modern mathematics.

Let in the functional equation

$$a_1\text{f}(g_1)+a_2\text{f}(g_2)+\ldots+a_n\text{f}(g_n)=b \qquad (A)$$

the expressions under the sign of the unknown function $f(x)$ are elements of the group G consisting of n functions: $g_1(x) = x, g_2(x), \dots, g_n(x)$, and the coefficients of equation (A) has a solution. Replace $x \to g_2(x)$. As a result, the sequence $g_1{}^\circ g_2, g_2{}^\circ g_2, \dots, g_n{}^\circ g_2$, again consisting of all elements of the group.

Therefore, the unknowns $\mathrm{f}(g_1), \mathrm{f}(g_2), \dots, \mathrm{f}(g_n)$ are interchanged and we get a new linear equation of the same form as (A). Further, in equation (A) we make changes $x \to g_3(x), x \to g_4(x), \dots, x \to g_n(x)$, after which we obtain a system of n linear equations, which should be solved. If there are solutions, then we still have to check to make sure that they satisfy equation (A).

As an example, consider the equation

$$2\mathrm{x}\mathrm{f}(x) + \mathrm{f}\left(\frac{1}{1-x}\right) = 2x \qquad (B)$$

The set of functions $g_1 = x, g_2 = \frac{1}{1-x}, g_3 = \frac{x-1}{x}$ forms a group with the multiplication table

0	g_1	g_2	g_3
g_1	g_1	g_2	g_3
g_2	g_2	g_3	g_1
g_3	g_3	g_1	g_2

Replacing in the equations (B) X by $\frac{1}{1-x}$, and by $\frac{x-1}{x}$, we obtain the system

$$2\mathrm{x}f_1 + f_2 = 2\mathrm{x},$$
$$\frac{2}{1-x}f_2 + f_3 = \frac{2}{1-x},$$
$$\frac{2(x-1)}{x}f_3 + f_1 = \frac{2(x-1)}{x},$$

where $f_1 = f(x), f_2 = f\big(g_2(x)\big) = f\left(\frac{x}{1-x}\right), f_3 = f\big(g_2(x)\big) = f\left(\frac{x-1}{x}\right)$, solving which we get by checking

$$f_1 = f(x) = \left(\frac{6x-2}{7x}\right) \text{ at } x \neq 0, x \neq -1$$

In conclusion, we present some examples of groups of functions that can be used to solve functional equations.

$$G_1 = \{x, a-x\}, G_2 = \left\{x, \frac{a}{x}\right\} \ (a \neq 0), G_3 = \left\{x, \frac{a}{x}, -x, -\frac{a}{x}\right\},$$
$$G_4 = \left\{x, \frac{1}{x}, -x, -\frac{-1}{x}, \frac{x-1}{x+1}, \frac{1-x}{1+x}, \frac{x+1}{x-1}, \frac{x+1}{1-x}\right\}$$
$$G_5 = \left\{x, \frac{a^2}{x}, a-x, \frac{ax}{x-a}, \frac{ax-a^2}{x}, \frac{a^2}{a-x}\right\}$$
$$G_6 = \left\{x, \frac{x\sqrt{3}-1}{x+\sqrt{3}}, \frac{x-\sqrt{3}}{x\sqrt{3}+1}, -\frac{1}{x}, \frac{x+\sqrt{3}}{1-x\sqrt{3}}, \frac{x\sqrt{3}+1}{\sqrt{3}-x}\right\}$$

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