A Note On Rings Of Real Valued Bc-Continuous Functions

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ABSTRACT

Some basis properties of bc-continuous functions are considered in this work. We establish and study the ring of all real valued bc-continuous functions on topological space $(\mathcal{M},\mathfrak{I})$ and denoted by $C_b(\mathcal{M})$.. It is proved that the ring of all real valued bc-continuous functions$C_b(\mathcal{M})$. is isomorphic to the $C_b(\mathcal{M})$. where$\mathcal{M}_{b^o}$ is b°-dimensional space

Keywords
b- open set, b°-dimensional space, b-compact space, bc-continuous functions, q-component

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Introduction

Throughout this paper, $(\mathcal{M},\mathfrak{I})$and $(\mathcal{N},\mathfrak{T})$ always represent a topological space and they will simply be written $\mathcal{M}$ and $\mathcal{N}$. Andrijevic[1] defined the concept of b-open sets as new class of generalized open sets. In 2013,Tahir [5] called function $\pi:(\mathcal{M},\mathfrak{I})\rightarrow(\mathcal{N},\mathfrak{T})$bc-continuous functionif for each $a\in\mathcal{M}$ and each open set G containing π(a), such that $\mathcal{N}-G$ is b-compact relative to $\mathcal{N}$, there exists an open set Fcontainingasuch that $\pi(F)\subseteq G$.

For a topological space $(\mathcal{M},\mathfrak{I})$, let $C(\mathcal{M},G)$denotes the ring of all continuous functions from $\mathcal{M}$ into G [6]. If G is the ring of real numbers $\mathbb{R}$, we will denote by $C(\mathcal{M})$ the ring of all continuous functions from $\mathcal{M}$ to $\mathbb{R}$. Moreover, $C_b(\mathcal{M})$ denotes to the set of real valued of bc-continuous functionon $\mathcal{M}$.Abjection function $i:(\mathcal{M},\mathfrak{I})\rightarrow(\mathcal{N},\mathfrak{T})$ is said to be b-homemorphism[6] if the function i and the inverse function i^{-1} are bc-continuous. Moreover, we call $(\mathcal{M},\mathfrak{I})$ and $(\mathcal{N},\mathfrak{T})$ are b-homemorphic and denoted by $\mathcal{M}\cong_b\mathcal{N}$.

Whenever $\kappa\in C(\mathcal{M})$, we will denote by Z(κ) to the zero setof κ, and Z(M)denotes to all zero-set Z(κ).Additionally, if $\pi\in C_b(\mathcal{M}), Z_b(\mathcal{M})$ denotes to all zero-sets Z_b (π).

We also need to recall the definition of the component.If $a\in\mathcal{M}$, the component C_a is the union of all connected subspaces of $(\mathcal{M},\mathfrak{I})$which contain a. Moreover, the q-componentQ_a of a point $a\in\mathcal{M}$is the intersection of all clopen (open and closed) subsets of a topological space $\mathcal{M}$ which contain a.

Recall that in [7] , A topological space $(\mathcal{M},\mathfrak{I})$ is said to be b-T_1-space if for every pair of distend points a andb in M, there exists a b-open set $H\subset\mathcal{M}$ such that $a\in H$ and $b\notin H$.

Preliminaries

Now, we introduce some basis definitions and facts of b-open sets [1], compactness [5], bc-continuous function.

Definition 2.1.

A subset F is said to be b-open set if $F\subseteq cl(int(F))\cup int(cl(F))$. The complement of a b-open set is b-closed , or equivalently , if $int(cl\ F)\cap cl(int\ F)\subseteq F$.

For a subset A of a space $(\mathcal{M},\mathfrak{I})$ the b-interior of A is the union of all b-open subsets of $\mathcal{M}$ and denoted by $int_b(A)$, andthe b-closure of A is the intersection of all b-closed subsets of A and denoted by $cl_b(A)$. A subset $S$of $\mathcal{M}$ is called $\mathscr{b}$ – open set if $S=int_b(S)$. In other words, A set S is $\mathscr{b}$ – open if and only if$S$ is a union of b- open set. Also, a set u is called $\mathscr{b}$- closed if and only if $U=cl_b(U)$ and a set U is $\mathscr{b}$- closed if and only if U is an intersection of b-closed set.Additionally, we will denote BO($\mathcal{M}$) to the family of all b-open in $(\mathcal{M},\mathfrak{I})$.

Definition 2.2.

A topological space $(\mathcal{M},\mathfrak{I})$ is said to be b-zero dimensional if and only if it is b-T_1-space and for each point $a\in\mathcal{M}$ and $a\notin B$ where B is closed subset of $\mathcal{M}$, there exists a b-open set A such that $A\cap B=\emptyset$. We will denote it by b°-dimensional

Or an equivalent definition a topological space $(\mathcal{M},\mathfrak{I})$ is called b°-dimensional if it is b-T_1-space and has a base containing of b-open set.

Definition 2.3.

Let F be a subset of a topological space $(\mathcal{M},\mathfrak{I})$, A cover $\{\mathfrak{w}_i | i \in \Omega\}$ of F by b-open sets of $\mathcal{M}$ is called b-open.

F is defined to be b-compact relative to $\mathcal{M}$ if every b-open cover of F has a finite subcover.

A space $(\mathcal{M}, \mathfrak{I})$ is defined to be b-compact if and only if M is b-compact relative to M.

Definition2.4.

A function π: $(\mathcal{M}, \mathfrak{I}) \rightarrow (\mathcal{N}, \mathfrak{T})$ is defined to be bc-continuous if for each $a \in \mathcal{M}$ and each open set G containing $\pi(a)$, such that $\mathcal{M} - G$ is b-compact relative to N, there exists an open set F containing a such that $\pi(F) \subseteq G$.

Definition 2.5.

A topological space $(\mathcal{M}, \mathfrak{I})$ is called b-Hausdorff if for pair of distinct points a and b in $\mathcal{M}$, there exists two b-open sets F and G such that $a \in F$, $b \in G$ and $F \cap G = \emptyset$.

Some properties of bc-continuous functions.

The results of this section will be extremely significant in the next section.

Proposition 3.1

.Every continuous function is bc-continuous.

Proof. $\alpha : (\mathcal{M}, \mathfrak{I}) \rightarrow (\mathcal{N}, \mathfrak{T})$ be a continuous function, and let $a \in \mathcal{M}$, G be an open set containing α (a)such that $\mathcal{M} - G$ is b-compact relative to $\mathcal{N}$.Since a$\in \mathcal{M}$, G is an open set containing α (a) and α is continuous,then there exists open set U containing a such that $\alpha(U) \subseteq G$. Then α is bc-continuous function. □

Remark 3.2.

The invers of the above proposition is not necessary to be true.

Two disjoint subsets F and G of space $(\mathcal{M}, \mathfrak{I})$ is said to be completely separated if there is a bc-continuous function $f \in C_b(\mathcal{M})$ that separates them.In addition, for the subset F and G such that $F \cap G = \emptyset$, then F and G be b-completely separated if and only if they are being in two disjoint members of $Z_b(\mathcal{M})$.

Proposition 3.3.

Let $(\mathcal{M}, \mathfrak{I})$ be a topological space and $(\mathcal{N}, \mathfrak{T})$ is b-compact space, if $\pi : (\mathcal{M}, \mathfrak{I}) \longrightarrow (\mathcal{N}, \mathfrak{T})$ be bc-continuous function, then the following statements hold.

If $a \in \mathcal{M}$,b in $\mathcal{N}$ and $\pi(a) = b$, then $\pi(Q_a) \subseteq Q_b$.

If N is a b-T_1-space, then π is constant on each q-componet in M .

Proof. 1) Let c in Q_a and space $\pi(c) \notin Q_b$. Then there is a closed and open set V in $\mathcal{N}$ such that $\pi(c) \in V$ and $V \cap Q_b = \emptyset$, that leads to $b \notin V$. since $\mathcal{N}$ is b-compact space and $\mathcal{N} - V$ is closed, then $\mathcal{N} - V$ is b- closed. So,$\mathcal{N} - V$ is b-compact relative to $\mathcal{N}$.Because of π is bc-continuous function, exists then there a b- open set H in $\mathcal{M}$ containing c such that $\pi(V) \subseteq H$. Also, $\pi(Q_b) \subseteq V$ (since $\pi(Q_a) \subseteq H$) and this implies to $\pi(a) = b$ in V, but this leads to the contradiction. □

2) Let $b \in Q_a$ and $\pi(a) \neq \pi(b)$, so there exsits an open set V in $\mathcal{N}$ such that $\pi(a)$ in V, but $\pi(b) \notin V$. $\mathcal{N} - V$ is b-compact relative in $\mathcal{N}$ (since $\mathcal{N}$ is b-compact space). Because π is bc-continuous, then there exists a b-open set H such that $\pi(H) \subseteq V$. However, any b-open set containing a, also contains b, for b in Q_a. So $\pi(b)$ in V, but that leads to contradiction. □

Corollary 3.4

If $(\mathcal{M}, \mathfrak{I})$ be a topological space, then for every $\pi \in C_b(\mathcal{M})$ the following cases holds.

1. On each q- component in $\mathcal{M}$, π is constant.
2. For $a \in \mathcal{M}$ the zero set $Z(\kappa) = \cup_{a \in Z(\kappa)} Q_a$

The ring $C_b(\mathcal{M}$

Theorem 4.1.

Whenever $(\mathcal{N}, \mathfrak{T})$ is b-compact, then $C(\mathcal{M}) = C_b(\mathcal{M})$.

Proof. The first implication $C(\mathcal{M}) \subset C_b(\mathcal{M})$ is clear.For another side, we should prove that $C_b(\mathcal{M}) \subset C(\mathcal{M})$. Let $\pi : (\mathcal{M}, \mathfrak{I}) \longrightarrow (\mathcal{N}, \mathfrak{T}) \in C_b(\mathcal{M})$, and let $a \in \mathcal{M}$, H is an open set containing $\pi(a)$. Indeed, $\mathcal{N} - H$ is closed,then $\mathcal{N} - H$ is b-compact (since $\mathcal{N}$ is b-compact). Because of π is bc-continuous then there exists an open set A containing a, such that $\pi(a) \subseteq H$. Hence $\pi : (\mathcal{M}, \mathfrak{I}) \longrightarrow (\mathcal{N}, \mathfrak{T})$ is continuous. □

Proposition 4.2.

Whenever $(\mathcal{M}, \mathfrak{I})$ is a topological space. If F is a b-closed subset of $\mathcal{M}$ and a in $\mathcal{M}$-F , then there exsits θ in $C_b(\mathcal{M})$ such that $\theta(F) = \{1\}$ and $\theta(Q_a) = \{0\}$.

Proof. Since $\mathcal{M} - F$ is b-open, then there exists a b-open set H containing a such that $H \cap F = \emptyset$.Now define an idempotent i with $Z(i) = H$. In other words,

$i(H) = \{0\}$ and $i(\mathcal{M} - H) = \{1\}$. It is easy to see that i in $C_b(\mathcal{M})$, $i(Q_a)$ =0 because of $Q_a \subset H$ and $i(F) = 1$. □

Proposition 4.3.

For a topological space $(\mathcal{M}, \mathfrak{I})$, $\mathcal{M}$ is b-Hausdorff if and only if for each $a \in \mathcal{M}$, $Q_a = \{a\}$.
Proof. Suppose that $\mathcal{M}$ is b-Hausdorff space and a, b in $\mathcal{M}$ such that $a \neq b$, then there exists a b-open set F containing a but not b. This leads to $b \notin Q_a$, i.e.,Q_a is single.
Conversely, let a and b in $\mathcal{M}$ such that $a \neq b$. So there exists a b-open set F such that $a \in F$ and $b \notin F$, i.e., $\mathcal{M}$ is Hausdorff.□
Next, we will explain that $C_b(\mathcal{M})$ is isomorphic with $C(\mathcal{N})$, where $\mathcal{N}$ is a b°-dimensional.

Theorem 4.4

.If $(\mathcal{M}, \mathfrak{I})$is a topological space, then there exists a b°-dimensional space $\mathcal{M}_{b^o}$ such that $C_b(\mathcal{M}) \cong C(\mathcal{M}_{b^o})$.
Proof. Suppose that Q_a is the q-compact of a, for each a in $\mathcal{M}$, and let $\mathcal{M}_{b^o}$ ={ $Q_a \mid a \in \mathcal{M}$ } is the decomposition. Moreover, we will consider τ is a topology on$\mathcal{M}_{b^o}$, so that H inτ if and only if $\bigcup_{Q_a \in H} Q_a$ is b-open in $\mathcal{M}$. It is easy to see that τ is a topology because $\mathcal{M} = \bigcup_{Q_a \in \mathcal{M}_{b^o}} Q_a$ and $\emptyset = \bigcup_{Q_a \in \emptyset} Q_a$ lead to $\mathcal{M}_{b^o}$ and Q are open. If A and B are open subsets of$\mathcal{M}_{b^o}$, then $\bigcup_{Q_a \in A \cap B} Q_a = (\bigcup_{Q_a \in A} Q_a) \cap (\bigcup_{Q_a \in B} Q_a)$leads to $A \cap B$ is open in $\mathcal{M}_{b^o}$. Also, it is clear for each A_1 in $\mathcal{M}_{b^o}$, $\bigcup_1 A_1$ is open in $\mathcal{M}_{b^o}$. Additionally, we can show that $\mathcal{M}_{b^o}$ is Hausdorff space, since if $Q_a$and Q_b in $\mathcal{M}_{b^o}$ such that $Q_a \cap Q_b = \emptyset$ and$a, b \in \mathcal{M}$, then $a \notin Q_b$and because Q_b is b-closed.So, from proposition [4.2]we can get an idempotent$i$ in $C_b(\mathcal{M})$, such that $i(Q_b) = 0$ and$i(Q_a) = 1$.
If we suppose that $K = \{Q_c | c \in Z(i)\}$, then $Z(i) = \bigcup_{Q_c \in K} Q_c$ leads to K is a b-open subset of $\mathcal{M}_{b^o}$.
In addition,$Q_b \in K$ but $Q_a \notin K$ in other words $\mathcal{M}_{b^o}$ is b-Hausdorff space. Therefore, it is Hausdorff.
Now, we need to explain $\mathcal{M}_{b^o}$ is $b^{\circ} - dimension$. Let K be closed set in $\mathcal{M}_{b^o}$ and $Q_b \notin K$.Therefore, $G = \bigcup_{Q_a \in K} Q_a$ is a b-closed subset of $\mathcal{M}$ and $b \notin G$. Hence, by proposition [4.3], there exists a b-open subset V of, such that $G \subset V$ and $b \notin V$.
$\bigcup_{c \in V} Q_c = V$ leads to $T = \{Q_c | c \in V\}$ is a b-open subset of V. It is clear that $K \subseteq T$ and $Q_b \notin K$, so $\mathcal{M}_{b^o}$ isb°-dimensional.
Now, we need to prove that $C_b(\mathcal{M}) \cong C(\mathcal{M}_{b^o})$. Hence we define $\pi : C_b(\mathcal{M}) \longrightarrow C(\mathcal{M}_{b^o})$ by $\pi(h) = h_{b^o}$for each $h \in C_b(\mathcal{M})$and $h_{b^o}$is define by $h_{b^o}(Q_a) = h(a)$ for each a in $\mathcal{M}$. By corollary (3.4), it is clear that $\pi$is well defined. Furthermore, $h_{b^o} \in C(\mathcal{M}_{b^o})$ for $h \in C_b(X)$. clearly if $h_{b^o}(Q_a) = f(a) = z$ then for each $\epsilon > 0$, there exsits a b-open subset K of $\mathcal{M}$ containing a such that $h(K) \subseteq (z - \epsilon, z + \epsilon)$leads to h_{b^o} in $C(\mathcal{M}_{b^o})$.
If $\pi(h) = \pi(f)$, where h, f in $C_b(\mathcal{M})$ then $h_{b^o} = f_{b^o}$ leads to h(a)=$h_{b^o}(Q_a) = f_{b^o}(Q_a) = f(a)$, for all a$\in \mathcal{M}$. So,$h$=$f$, in other words, π is one-to-one.

Moreover, it is easy to see that π is homomorphism because $\pi(h + f) = (h + f)_{b^o}$ and $(h + f)_{b^o}(Q_a) = (h + f)(a) = h(a) + f(a) = h_{b^o}(a)$ $+ f_{b^o}(a)$ for each Q_a in $\mathcal{M}_{b^o}$. And this leads to $\pi(h + f) = \pi(h) + \pi(f)$, for all h, f in $C_b(\mathcal{M})$, therefore,π is homomorphism.
Now, we will show that π is onto. suppose that $f \in C(\mathcal{M}_{b^o})$ and we define $h: \mathcal{M} \longrightarrow \mathbb{R}$ by $h(a) = f(Q_a)$for all a in $\mathcal{M}$ and it is bc-continiuous.
Clearly, if$a \in \mathcal{M}$, $h(a) = h(Q_a) = z$ and let $\epsilon > 0$ is given, then there is an open subset G of$\mathcal{M}_{b^o}$ containing Q_a such that $f(G) \subseteq (z - \epsilon, z + \epsilon)$. We can see that is enough to consider G-open subset $H = \bigcup_{Q_z \in H} Q_zof\mathcal{M}$.
If fact, $a \in H$and $h(H) \subseteq (z - \epsilon, z + \epsilon)$ which leads to in$C_b(\mathcal{M})$. It has been seen from definition of h andπ that $\pi(h) = f$. Therefore, we have proved that π is onto.□

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