Order Statistical and Censored Data For Weibull Lindley Rayleigh Distribution

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Abstract.
In this paper, we study statistical inferences on the maximum likelihood estimation of the Weibull Lindley Rayleigh distribution when data are randomly censored. Likelihood equations are derived assuming that the censoring distribution does not involve any parameters of interest. The maximum likelihood estimators (MLEs) of the censored the Weibull Lindley Rayleigh distribution do not have an explicit form, and it should be solved in an iterative way. By using the same method, the observed Fisher information is also approximated to obtain asymptotic variances of the estimators. An illustrative example is presented, and a simulation study is conducted to compare the performances of the estimators. In addition to their explicit form, the approximate MLEs are as efficient as the MLEs in terms of variances.

Keywords:
Weibull Lindley Rayleigh distribution; Maximum likelihood estimators; Simulation, random censoring.

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1. INTRODACTION

In statistical analysis of life time data, some well-known common distributions are exponential,Weibull, lognormal, and gamma distribution. The lognormal distribution is especially useful when a hazard rate is initially increasing and then decreasing. We also need inferences for a normal distributionsince the logarithm of a lognormal variable follows a normal. As for censoring types, the most common and simplest censoring schemes are type I or type II censoring. For numerous censoring types, see [30], and [25].

[16], [12], [13], and [22] studied the estimation of a type II censored normal distribution. [6] dealt with the estimation of a normal distribution on a progressively type II censoring scheme, which is a generalization of traditional type II censoring. Maximum likelihood estimators (MLEs) based on type II or progressive type II censored data from a normal distribution do not have explicit forms, and the situation remains as it is for many distributions when data are censored. [21] and [6] derived approximate MLEs under type II censoring and progressively type II censoring of a normal, respectively.

The approximation method was first developed [3] to find the approximate MLE of the scalar parameter in the Rayleigh distribution with left and right type II censoring. The method approximates the nonlinear part of the likelihood equations; subsequently, many researchers used it for other distributions under several censoring schemes that are most often progressively type II censoring. [6] did it for a normal distribution. [5] , and [7] studied the estimation for the logistic distribution and the extreme value distribution, respectively. [1] and [2] dealt with the problem for generalized logistic distribution and generalized exponential distribution, respectively. [4] used the method for the half logistic distribution. [28] , and [18] discussed the procedure for the Rayleigh distribution. [29] used it for inverse Weibull distribution. As for random censoring, [20] and [22] applied the approximate method to generalized exponential distributions and Weibull distribution, respectively. In this paper, we study the order statistics of Weibull Lindley Rayleigh distribution (WLRD) and maximum likelihood estimators based on type I censored data.

In section 2, In this section, we study the order statistics of WLRD.

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In section 3, we derive MLEs for the Weibull Lindley Rayleigh distribution under type I censored data. In section 4, we presents simulation results by computing the bias and mean square error of parameters under censored data. Finally, we give our conclusions.

The pdf of Weibull Lindley Rayleigh distribution [31] is

$$h_{WLR}(x)=\frac{2\theta^2 x}{\lambda+1}\left[\lambda\beta\left(1-\exp(-\theta^2x^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2x^2)\right)\right]\times\exp\left(-(\lambda+\beta)(1-exp(-\theta^2x^2))\right)\exp(-\theta^2x^2) \tag{1}$$

And cdf is

$$H_{WLR}(x)=1-\frac{1+\lambda+\lambda(1-\exp(-\theta^2x^2))}{\lambda+1}\exp\left(-(\lambda+\beta)(1-\exp(-\theta^2x^2))\right) \tag{2}$$

The joint pdf of $Y_1, Y_2, \dots, Y_n$ is given by.

$$g_{i:n}(y_i)=\begin{cases}\dfrac{n!}{(i-1)!\,(n-i)!}[F(y_i)]^{i-1}[1-F(y_i)]^{n-i}f(y_i) & \text{if } a<y_i<b,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Where $Y_1<Y_2<\cdots<Y_n$.

2 ORDER STATISTICS OF WEIBULL LINDLEY RAYEIGH DISTRIBUTION

In this section, we derive the joint pdf of order statistics of WLRD. The r-moments are calculated. Let Y_1 be the smallest of these X_i, Y_2 the next X_i in order of magnitude,..., and Y_n the largest of X_i. That is, $Y_1<Y_2<\cdots<Y_n$ represent $X_1, X_2, \dots, X_n$ when the latter are arranged in ascending order of magnitude. We call $Y_i, i=1,2,\dots,n$, the i th order statistic of the random sample $X_1, X_2, \dots, X_n$.

Then the joint pdf of WLRD is given by

$$\begin{aligned}h_{WLRDi:n}(y_i,\eta)&=\frac{n!}{(i-1)!\,(n-i)!}(H_{WLRD}(y_i,\eta))^{i-1}\left(1-H_{WLRD}(y_i,\eta)\right)^{n-i}h_{WLRD}(y_i,\eta)\\&=\frac{n!}{(i-1)!\,(n-i)!}\left[1-\frac{\tau_1+\lambda\left(1-\exp(-\theta^2y_i^2)\right)}{\tau_1}\exp\left(-\tau_3\left(1-\exp(-\theta^2y_i^2)\right)\right)\right]^{i-1}\\&\times\left[\frac{\tau_1+\lambda(1-\exp(-\theta^2y_i^2))}{\tau_1}\exp\left(-\tau_3(1-\exp(-\theta^2y_i^2))\right)\right]^{n-i}\\&\times\left[\frac{2\theta y_i}{\tau_1}\left(\tau_2\left(1-\exp(-\theta^2y_i^2)\right)+(\tau_2+\beta)\right)\right]\exp(-\tau_3(1-exp(-\theta^2y_i^2))-\theta^2y_i^2)\end{aligned} \tag{4}$$

Where $\tau_1=\lambda+1, \tau_2=(\beta\lambda+\lambda^2), \tau_3=\lambda+\beta$

Then we get,

$$\begin{aligned}h_{WLRDi:n}(y_i,\eta)&=\frac{n!}{(i-1)!\,(n-i)!}\left[1-\frac{\tau_1+\lambda(1-\exp(-\theta^2y_i^2))}{\tau_1}\exp\left(-\tau_3(1-\exp(-\theta^2y_i^2))\right)\right]^{(i-1)}\\&\times\left[\frac{\tau_1+\lambda\left(1-\exp(-\theta^2y_i^2)\right)}{\tau_1}\right]^{(n-i)}\\&\times\left[\frac{2\theta^2y_i}{\tau_1}\left(\tau_2\left(1-\exp(-\theta^2y_i^2)\right)+(\tau_2+\beta)\right)\right]\\&\times\exp(-\tau_3(n-i+1)(1-\exp(-\theta^2y_i^2))-\theta^2y_i^2)\end{aligned} \tag{5}$$

then

$$h_{WLRDi:n}(y_i,\eta)=\frac{n!}{(i-1)!\,(n-i)!}\left[\frac{\tau_1}{\tau_1}-\frac{\tau_1+\lambda\left(1-\exp(-\theta^2y_i^2)\right)}{\tau_1}\exp\left(-\tau_3\left(1-\exp(-\theta^2y_i^2)\right)\right)\right]^{(i-1)}$$

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$$\times\left[\frac{\tau_1+\lambda\left(1-\exp(-\theta^2 y_i^2)\right)}{\tau_1}\right]^{(n-i)}\times\left[\frac{2\theta^2 y_i}{\tau_1}\left(\tau_2-\tau_2\exp(-\theta^2 y_i^2)\right)+(\tau_2+\beta)\right)$$
$$\times\exp(-\tau_3(n-i+1)(1-\exp(-\theta^2 y_i^2))-\theta^2 y_i^2$$
$$=\frac{n!}{(i-1)!\,(n-i)!\,\tau_1^{i-1}}\left[\tau_1-\left(\tau_1+\lambda\left(1-\exp(-\theta^2 y_i^2)\right)\right)\exp\left(-\tau_3\left(1-\exp(-\theta^2 y_i^2)\right)\right)\right]^{(i-1)}$$
$$\times\frac{1}{\tau_1^{n-i}}\left[\tau_1+\lambda\left(1-\exp(-\theta^2 y_i^2)\right)\right]^{(n-i)}\left[\frac{2\theta^2 y_i}{\tau_1}\left(\tau_2-\tau_2\exp(-\theta^2 y_i^2)\right)+(\tau_2+\beta)\right)$$
$$\times\exp(-\tau_3(n-i+1)(1-\exp(-\theta^2 y_i^2))-\theta^2 y_i^2 \qquad (6)$$

Implies that

$$h_{WLRDi:n}(y_i,\eta)=\frac{2\theta^2 n!\,y_i}{(i-1)!\,(n-i)!\,\tau_1^n}\left[\tau_1-\left(\tau_1+\lambda\left(1-\exp(-\theta^2 y_i^2)\right)\right)\exp\left(-\tau_3\left(1-\exp(-\theta^2 y_i^2)\right)\right)\right]^{(i-1)}$$
$$\times\left[\tau_1+\lambda\left(1-\exp(-\theta^2 y_i^2)\right)\right]^{(n-i)}\left[(2\tau_2+\beta)-\tau_2\exp(-\theta^2 y_i^2)\right]$$
$$\times\exp(-\tau_3(n-i+1)(1-exp(-\theta^2 y_i^2))\,exp(-\theta^2 y_i^2) \qquad (7)$$

The PDF of the smallest order statistic y_1 is:

$$h_{WLRD1:n}(y_1,\eta)=\frac{2\theta^2 n y_1}{\tau_1^n}\left[\tau_1+\lambda\left(1-\exp(-\theta^2 y_1^2)\right)\right]^{(n-1)}\left[(2\tau_2+\beta)-\tau_2\exp(-\theta^2 y_1^2)\right]$$
$$\times\exp(-\tau_3(n)(1-exp(-\theta^2 y_1^2))\,exp(-\theta^2 y_1^2) \qquad (8)$$

The PDF of the largest order statistics y_n is:

$$h_{WLRDn:n}(y_n,\eta)=\frac{2\theta^2 n y_n}{\tau_1^n}\left[\tau_1-\left(\tau_1+\lambda\left(1-\exp(-\theta^2 y_n^2)\right)\right)\exp\left(-\tau_3\left(1-\exp(-\theta^2 y_n^2)\right)\right)\right]^{(n-1)}$$
$$\times\left[(2\tau_2+\beta)-\tau_2\exp(-\theta^2 y_n^2)\right]$$
$$\times\exp(-\tau_3(1-exp(-\theta^2 y_n^2))\,exp(-\theta^2 y_n^2) \qquad (9)$$

2.1 Moment Of Order Statistics: By making some simplifications and mathematical calculations, applying a Tyler expansion, and the law of moment of order statistics we get,

$$E(y_i^r)=\int_0^\infty\frac{2\theta^2 y_i^{r+1}\exp(-\tau_3)n!}{\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{j=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$
$$\times\left[(2\tau_2+\beta)-\tau_2 exp(-\theta^2 y_i^2)\right]exp(-\theta^2 y_i^2(k+\ell+1))dy_i$$
$$=\frac{2\theta^2 exp(-\tau_3)n!}{\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$
$$\times\left[\int_0^\infty y_i^{r+1}\left[(2\tau_2+\beta)exp(-\theta^2 y_i^2(k+\ell+1))-\tau_2 exp(-\theta^2 y_i^2(k+\ell+2))\right]dy_i\right]$$
$$=\frac{2\theta^2 exp(-\tau_3)n!}{\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$
$$\times\left[\int_0^\infty y_i^{r+1}\left[(2\tau_2+\beta)exp(-\theta^2 y_i^2(k+\ell+1))-\tau_2 exp(-\theta^2 y_i^2(k+\ell+2))\right]dy_i\right]$$
$$=\frac{2\theta^2 exp(-\tau_3)n!}{\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$
$$\times\left[(2\tau_2+\beta)\int_0^\infty y_i^{r+1}exp(-\theta^2 y_i^2(k+\ell+1))dy_i-\tau_2\int_0^\infty y_i^{r+1}exp(-\theta^2 y_i^2(k+\ell+2))dy_i\right]$$

let $-\theta^2 y_i^2(k+\ell+1)=p\Rightarrow y_i=\frac{\sqrt{p}}{\theta\sqrt{\ell+k+1}},dy_i=\frac{-dp}{2\theta\sqrt{\ell+k+1}\sqrt{p}}$

and let $-\theta^2 y_i^2(k+\ell+2)=q\Rightarrow y_i=\frac{\sqrt{q}}{\theta\sqrt{\ell+k+2}},dy_i=\frac{-dq}{2\theta\sqrt{\ell+k+2}\sqrt{q}}$

$$E(y_i^r)=\frac{2\theta^2 exp(-\tau_3)n!}{\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

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$$\times\left[(2\tau_2+\beta)\int_0^\infty \frac{p^{\frac{r+1}{2}}exp(-p)}{\theta^{r+1}(\ell+k+1)^{\frac{r+1}{2}}}\cdot\frac{dp}{2\theta\sqrt{\theta+k+1}\sqrt{p}}-\tau_2\int_0^\infty \frac{q^{\frac{r+1}{2}}exp(-q)}{\theta^{r+1}(\ell+k+2)^{\frac{r+1}{2}}}\cdot\frac{dq}{2\theta\sqrt{\theta+k+2}\sqrt{q}}\right]$$

$$=\frac{exp(-\tau_3)n!}{\theta^r\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^{\frac{r+1}{2}}}\int_0^\infty p^{\frac{r}{2}}\exp(-p)dp-\frac{\tau_2}{(\ell+k+2)^{\frac{r+1}{2}}}\int_0^\infty q^{\frac{r}{2}}\exp(-p)dp\right]$$

$$=\frac{\exp(-\tau_3)n!}{\theta^r\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1\right)-\frac{\tau_2}{(\ell+k+2)^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1\right)\right]$$

Where $\Gamma\left(\frac{r}{2}+1\right)$ is gamma function

Then we get

$$E(y_i^r)=\frac{\exp(-\tau_3)n!}{\theta^r\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^{\frac{r}{2}+1}}-\frac{\tau_2}{(\ell+k+2)^{\frac{r}{2}+1}}\right]\Gamma\left(\frac{r}{2}+1\right)\qquad(10)$$

one can find the mean of of order statistics WLRD by putting r=1 we get

$$E(y_i)=\frac{\exp(-\tau_3)n!}{\theta\,\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^{\frac{1}{2}+1}}-\frac{\tau_2}{(\ell+k+2)^{\frac{1}{2}+1}}\right]\Gamma\left(\frac{1}{2}+1\right)\qquad(11)$$

at r=2 we obtain the second moment is

$$E({y_i}^2)=\frac{\exp(-\tau_3)n!}{\theta^2\,\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times 2\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^2}-\frac{\tau_2}{(\ell+k+2)^2}\right]\qquad(12)$$

Then the variance is

$$V\,(y_i)=\frac{\exp(-\tau_3)n!}{\theta^2\,\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}$$

$$\times 2\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^2}-\frac{\tau_2}{(\ell+k+2)^2}\right]$$

$$-\left[\frac{\exp(-\tau_3)n!}{\theta\,\tau_1(i-1)!\,(n-i)}\sum_{j=0}^\infty(-1)^j\binom{i-1}{j}\sum_{m=0}^\infty\binom{j+n-1}{m}\frac{\lambda^m}{\tau_1^m}\sum_{\ell=0}^\infty\binom{m}{\ell}(-1)^\ell\sum_{k=0}^\infty(-1)^k\frac{\tau^k}{k!}\right.$$

$$\left.\times\left[\frac{(2\tau_2+\beta)}{(\ell+k+1)^{\frac{1}{2}+1}}-\frac{\tau_2}{(\ell+k+2)^{\frac{1}{2}+1}}\right]\Gamma\left(\frac{1}{2}+1\right)\right]^2\qquad(13)$$

3 CENSORED DATA OF WLRD In this section, we find the maximum likelihood estimator based on type I censored data. The ikelihood function is

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$$L(t,\theta,\beta,\lambda)=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{i}h\left(t_{(i)}\right)\right]\left[S\left(t_{(0)}\right)\right]^{n-r},0\le t_{(1)}\le\cdots\le t_{(0)}<\infty$$
$$=\frac{n!}{(n-r)!}\left[\frac{2^r\theta^{2r}\prod_{i=1}^{r}t_i}{(\lambda+1)^r}\right]\left[\prod_{i=1}^{r}\left(\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(\left(2-\exp(-\theta^2t_i^2)\right)\right)\right)\right]$$
$$\times\exp\left(-(\lambda+\beta)\sum_{i=1}^{r}\left(1-\exp(-\theta^2t_i^2)\right)-\theta^2t_i^2\right)$$
$$\times\left[\frac{1+\lambda+\lambda\left(\left(1-\exp(-\theta^2t_0^2)\right)\right)}{\lambda+1}\exp\left(-(\lambda+\beta)\left(1-\exp(-\theta^2t_0^2)\right)\right)\right]^{n-r} \tag{14}$$

$$\begin{aligned}LogL(t,\theta,\beta,\lambda)= &\log\frac{n!}{(n-r)!}+r\log 2+2r\log\theta+\sum_{i=1}^{r}\log t_i-r\log(\lambda+1)\\&+\sum_{i=1}^{r}\log\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(\left(2-\exp(-\theta^2t_i^2)\right)\right)\right]\\&-(\lambda+\beta)\sum_{i=1}^{r}\left(1-\exp(-\theta^2t_i^2)\right)-\theta^2\sum_{i=1}^{r}t_i^2\\&+(n-r)\left[\log\left(1+\lambda+\lambda(1-\exp(-\theta^2t_0^2)\right)-\log(\lambda+1)-(\lambda+\beta)(1-\exp(-\theta^2t_0^2)))\right]\end{aligned} \tag{15}$$

The likelihood equations are

$$\begin{aligned}U_\theta=0=\ &\frac{2r}{\theta}+\sum_{i=1}^{r}\frac{2\theta\lambda\beta t_i^2\exp(-\theta^2t_i^2)+2\theta\lambda^2t_i^2\exp(-\theta^2t_i^2)}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]}\\&-2\theta(\lambda+\beta)\sum_{i=1}^{r}t_i^2\exp(-\theta^2t_i^2)-2\theta\sum_{i=1}^{r}t_i^2\\&-(n-r)\left[\frac{2\theta\lambda t_0^2\exp(-\theta^2t_0^2)}{1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))}-2\theta(\lambda+\beta)t_0^2\exp(-\theta^2t_0^2)\right]\end{aligned} \tag{16}$$

$$\begin{aligned}U_\beta=0=\ &\sum_{i=1}^{r}\frac{\lambda\left(1-\exp(-\theta^2t_i^2)\right)+(\lambda+1)}{\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)}-\sum_{i=1}^{r}\left(1-\exp(-\theta^2t_i^2)\right)\\&-(n-r)(1-\exp(-\theta^2t_0^2))\end{aligned} \tag{17}$$

$$\begin{aligned}U_\lambda=0=&-\frac{r}{(\lambda-1)}+\sum_{i=1}^{r}\frac{\beta(1-\exp(-\theta^2t_i^2))+\beta+2\lambda(2-\exp(-\theta^2t_i^2))}{[\lambda\beta(1-\exp(-\theta^2t_i^2))+\beta(\lambda+1)+\lambda^2(2-\exp(-\theta^2t_i^2))]}\\&-\sum_{i=1}^{r}(1-\exp(-\theta^2t_i^2))\\&+(n-r)\left[\frac{(2-\exp(-\theta^2t_0^2))}{1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))}-\frac{1}{\lambda+1}-(1-\exp(-\theta^2t_0^2))\right]\end{aligned} \tag{18}$$

We obtain MLEs by solving the above equations numerically.

We consider the observed Fisher information matrix i of WLRD is given by using the above equations

$$i(\theta,\beta,\lambda)=\begin{pmatrix}U_{\theta\theta}&U_{\theta\beta}&U_{\theta\lambda}\\U_{\beta\theta}&U_{\beta\beta}&U_{\lambda\lambda}\\U_{\lambda\theta}&U_{\lambda\beta}&U_{\lambda\lambda}\end{pmatrix} \tag{19}$$

Where

$$U_{\theta\theta}=\frac{-2r}{\theta^2}++2\theta(\beta+\lambda)\sum_{i=1}^{r}t_i^2\exp(-\theta^2t_i^2)-2\sum_{i=1}^{r}t_i^2$$
$$+\sum_{i=1}^{r}\frac{[2t_i^2\lambda(\lambda+\beta)(1-2\theta^2t_i^2)\exp(-\theta^2t_i^2)]}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]}$$

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$$-\sum_{i=1}^{r}\frac{\left[(\beta+\lambda)^2\left(4\theta^2\lambda^2t_i^4\exp(-2\theta^2t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}$$
$$-(n-r)\left[\frac{-\lambda t_0^4\exp(-\theta^2t_0^2)[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]-[\lambda t_0^2\exp(-\theta^2t_0^2)]^2}{[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]^2}\right] \quad (20)$$

$$U_{\theta\beta}=\sum_{i=1}^{r}\frac{\left[\left(2\theta\lambda t_i^2\exp(-\theta^2t_i^2)\right)\left(\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}$$
$$+\sum_{i=1}^{r}\frac{(\lambda+\beta)\left(2\theta\lambda t_i^2\exp(-\theta^2t_i^2)\right)\left(\lambda\left(1-\exp(-\theta^2t_i^2)\right)+(\lambda+1)\right)}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}-\sum_{i=1}^{r}t_i^2\exp(-\theta^2t_i^2)$$
$$-(n-r)\left[\frac{t_0^2\exp(-\theta^2t_0^2)[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]-[\lambda t_0^2\exp(-\theta^2t_0^2)][2-\exp(-\theta^2t_0^2)]}{\left[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))\right]^2}\right]$$
$$-2\theta t_0^2\exp(-\theta^2t_i^2) \quad (21)$$

$$U_{\theta\lambda}=\sum_{i=1}^{r}\frac{\left[(\beta+2\lambda)\left(2\theta t_i^2\exp(-\theta^2t_i^2)\right)\left(\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}$$
$$+\sum_{i=1}^{r}\frac{(\lambda+\beta)\left(2\theta\lambda t_i^2\exp(-\theta^2t_i^2)\right)\left(\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta+2\lambda\left(2-\exp(-\theta^2t_i^2)\right)\right)}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}$$
$$-\sum_{i=1}^{r}t_i^2\exp(-\theta^2t_i^2)$$
$$-(n-r)\left[\frac{2\theta t_0^2\exp(-\theta^2t_0^2)[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]-[2\theta\lambda t_0^2\exp(-\theta^2t_0^2)][2-\exp(-\theta^2t_0^2)]}{[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]^2}\right]$$
$$-2\theta t_0^2\exp(-\theta^2t_i^2) \quad (22)$$

$$U_{\beta\theta}=\sum_{i=1}^{r}\frac{\left[\left(2\theta\lambda t_i^2\exp(-\theta^2t_i^2)\right)\left(\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}$$
$$+\sum_{i=1}^{r}\frac{(\lambda+\beta)\left(2\theta\lambda t_i^2\exp(-\theta^2t_i^2)\right)\left(\lambda\left(1-\exp(-\theta^2t_i^2)\right)+(\lambda+1)\right)}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2}-\sum_{i=1}^{r}t_i^2\exp(-\theta^2t_i^2)$$
$$-(n-r)\left[\frac{t_0^2\exp(-\theta^2t_0^2)[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))]-[\lambda t_0^2\exp(-\theta^2t_0^2)][2-\exp(-\theta^2t_0^2)]}{\left[1+\lambda+\lambda(1-\exp(-\theta^2t_0^2))\right]^2}\right]$$
$$-2\theta t_0^2\exp(-\theta^2t_i^2) \quad (23)$$

$$U_{\beta\beta}=\sum_{i=1}^{r}\frac{-\left[\lambda\left(1-\exp(-\theta^2t_i^2)\right)+(\lambda+1)\left[\lambda\left(1-\exp(-\theta^2t_i^2)\right)+(\lambda+1)\right]\right.}{\left[\lambda\beta\left(1-\exp(-\theta^2t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2t_i^2)\right)\right]^2} \quad (24)$$

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$$U_{\beta\lambda} = \sum_{i=1}^{r} \frac{\left[\left(1-\exp(-\theta^2 t_i^2)\right)+1\right]\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}$$
$$-\sum_{i=1}^{r} \frac{[\lambda(1-\exp(-\theta^2 t_i^2))+(\lambda+1)][\beta(1-\exp(-\theta^2 t_i^2))+\beta+2\lambda(2-\exp(-\theta^2 t_i^2))]}{[\lambda\beta(1-\exp(-\theta^2 t_i^2))+\beta(\lambda+1)+\lambda^2(2-\exp(-\theta^2 t_i^2))]^2} \tag{25}$$

$$U_{\lambda\theta} = \sum_{i=1}^{r} \frac{\left[(\beta+2\lambda)\left(2\theta t_i^2\exp(-\theta^2 t_i^2)\right)\left(\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}$$
$$+\sum_{i=1}^{r} \frac{(\lambda+\beta)\left(2\theta\lambda t_i^2\exp(-\theta^2 t_i^2)\right)\left(\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta+2\lambda\left(2-\exp(-\theta^2 t_i^2)\right)\right)}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}$$
$$-\sum_{i=1}^{r} t_i^2\exp(-\theta^2 t_i^2)$$
$$-(n-r)\left[\frac{2\theta t_0^2\exp(-\theta^2 t_0^2)[1+\lambda+\lambda(1-\exp(-\theta^2 t_0^2))]-[2\theta\lambda t_0^2\exp(-\theta^2 t_0^2)][2-\exp(-\theta^2 t_0^2)]}{\left[1+\lambda+\lambda(1-\exp(-\theta^2 t_0^2))\right]^2}\right]$$
$$-2\theta t_0^2\exp(-\theta^2 t_i^2) \tag{26}$$

$$U_{\lambda\beta} = \sum_{i=1}^{r} \frac{\left[\left(1-\exp(-\theta^2 t_i^2)\right)+1\right]\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}$$
$$-\sum_{i=1}^{r} \frac{[\lambda(1-\exp(-\theta^2 t_i^2))+(\lambda+1)][\beta(1-\exp(-\theta^2 t_i^2))+\beta+2\lambda(2-\exp(-\theta^2 t_i^2))]}{[\lambda\beta(1-\exp(-\theta^2 t_i^2))+\beta(\lambda+1)+\lambda^2(2-\exp(-\theta^2 t_i^2))]^2} \tag{27}$$

$$U_{\lambda\lambda} = \sum_{i=1}^{r} \frac{\left[2\left(2-\exp(-\theta^2 t_i^2)\right)\right]\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}$$
$$-\sum_{i=1}^{r} \frac{\left[\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta+2\lambda\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}{\left[\lambda\beta\left(1-\exp(-\theta^2 t_i^2)\right)+\beta(\lambda+1)+\lambda^2\left(2-\exp(-\theta^2 t_i^2)\right)\right]^2}+\frac{r}{(\lambda+1)^2}$$
$$-(n-r)\left[\frac{[2-\exp(-\theta^2 t_0^2)]^2}{[1+\lambda+\lambda(1-\exp(-\theta^2 t_0^2))]^2}\right]+\frac{1}{(\lambda+1)^2} \tag{28}$$

The Fisher information matrix I of WLRD is given by

$$I(\theta,\beta,\lambda) = -E\left(i(\theta,\beta,\lambda)\right) = -\begin{pmatrix} E(U_{\theta\theta}) & E(U_{\theta\beta}) & E(U_{\theta\lambda}) \\ E(U_{\beta\theta}) & E(U_{\beta\beta}) & E(U_{\beta\lambda}) \\ E(U_{\lambda\theta}) & E(U_{\lambda\beta}) & E(U_{\lambda\lambda}) \end{pmatrix} \tag{29}$$

4 SIMULATION

The simulation of WLRD is assessing the performance of the maximum likelihood estimators given by equations (16), equation (17) and equation (18) with respect to sample size r. We compute the maximum likelihood estimators for the ten thousand samples, say $\hat{\beta}_i, \hat{\lambda}_i, \hat{\theta}_i$ for $i = 1,2,\dots,10000$. Finally, we calculate the biases and mean squared errors given by

$$\text{bias}_h(r) = \frac{1}{10000}\sum_{i}^{10000}\left(\hat{h}_i - h_i\right) \tag{30}$$

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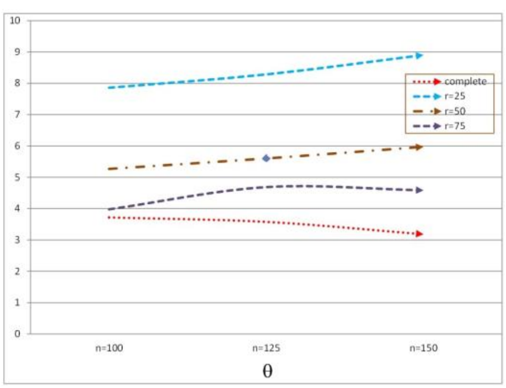
and

$$MSE_h(r) = \frac{1}{10000}\sum_{i}^{10000}\left(\hat{h}_i - h_i\right)^2 \qquad (31)$$

where $h = \beta, \lambda, \theta$ under assumption n = 100,125,150. And r=25,50,75 we get the table 1 and the plot of bias and MSE of β, λ and θ with respect to WLRD(β, λ, θ) respectively in figure A, figure B, figure C, figure D figure E and figure F.

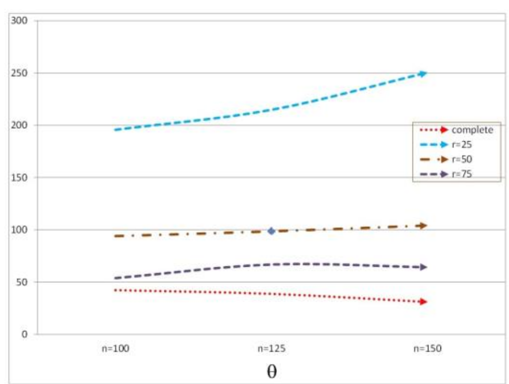
Tabal (1) mean square error and bias for estimates based on censored data type I

			Estimate			**Bias**			**MSE**		
n	□□□□□□□□□ □	**r**	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$
100	200 , 200 , 200	100	203.723	200.9745	200.974	3.722892	0.974471	0.974449	42.4139	4.3475	4.34747
		25	207.863	201.9741	201.974	7.863429	1.974056	1.974007	195.774	12.374	12.3732
		50	205.273	201.3235	201.324	5.272691	1.323493	1.323461	94.1754	5.9385	5.93816
		75	203.982	200.9999	201	3.981827	0.999906	0.999881	53.9699	3.4071	3.40692
125	200 , 200 , 200	125	203.583	200.8963	200.896	3.583285	0.89625	0.896228	38.9101	2.4505	2.45043
		25	208.987	202.2549	202.255	8.287188	2.054928	2.054872	214.814	13.921	13.9205
		50	205.604	201.418	201.418	5.604273	1.417951	1.417917	98.7349	6.3514	6.35109
		75	204.693	201.1788	201.179	4.692519	1.178792	1.178763	66.993	4.2325	4.23227
150	200 , 200 , 200	150	203.19	200.803	200.803	3.189952	0.803003	0.802983	31.1419	1.9811	1.98101
		25	208.207	202.0589	202.059	8.906523	2.258863	2.258812	250.438	15.496	15.4954
		50	205.98	201.5008	201.501	5.979803	1.500762	1.500725	104.315	6.5744	6.57407
		75	204.588	201.1723	201.172	4.588209	1.17234	1.172312	64.398	4.4704	4.47019



(A)

FIGURE(A) The graph of bias for θ



(B)

FIGURE(B) The graph of mean square error for θ

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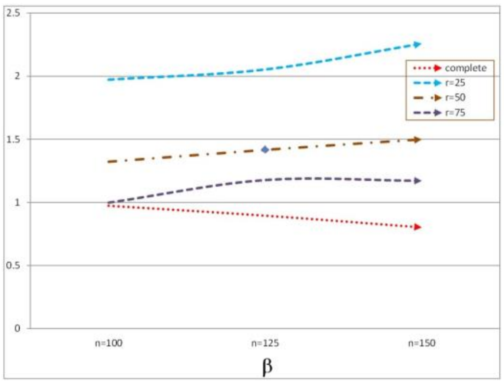
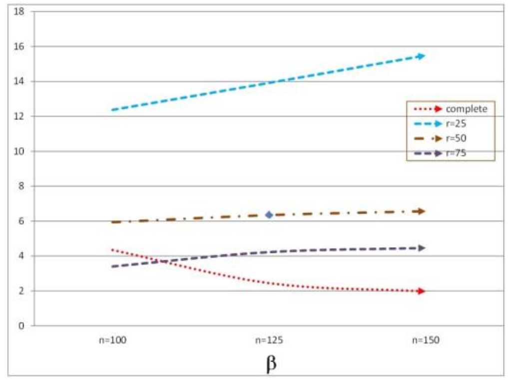


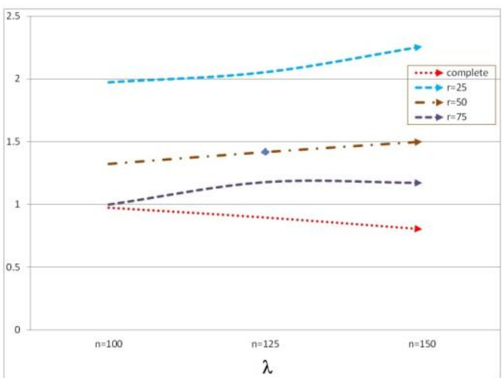
(C)

FIGURE (C) The graph of bias for β



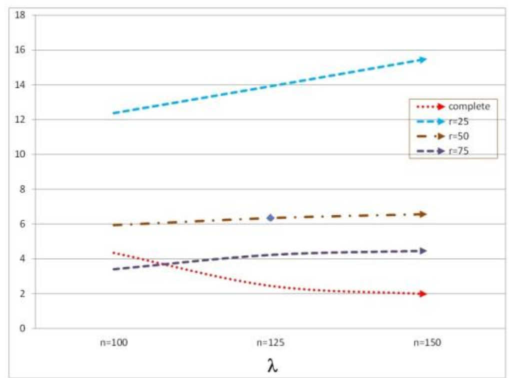
(D)

FIGURE(D) The mean square error for β



(E)

FIGURE(E) The graph of bias for λ



(F)

FIGURE(F) The graph of mean square error for λ

5 CONCLUSION

In this paper, we derive joint pdf of order statistics of Weibull Lindley Rayleigh distribution, in this side, the r-moments are driven. The maximum likelihood estimators of WLRD based on censored data. We solve the maximum likelihood equations numerically to obtain the maximum estimators. We make simulation by drawing the bias and mean square error for WLRD for every estimators.

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