

Mathematical Concept Construction through Abstraction: in the View of APOS and AiC Theory

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ABSTRACT

The cognition study about understanding mathematical concepts brings to abstraction, which is transforming mathematical knowledge into new concepts on the higher level. Although there are several theories on abstraction, the theory of Abstraction in Context (AiC) and APOS are the two most widely studied and comprehensively developed. This paper examines the abstraction, in the view of APOS theory and AiC, since it plays an essential role in forming mathematical knowledge. The research methods is literature, and the results show that both of these theory has developed from fundamental epistemology to the abstraction process, pedagogical implementation, and research methods. Exploration and comparison to APOS and AiC showed that both theories provide a robust theoretical foundation for constructors. The abstraction process in APOS theory occurs through mental mechanisms that give schema as the final result, while AiC goes through RBC epistemic actions that produce the construct. Both APOS and AiC agreed that abstraction does not appear by itself, and they suggested group learning to foster abstraction. In particular, APOS recommends the ACE strategy. APOS and AiC also provide research methods to guide educators and researchers working with abstractions. Both of their approaches need the identification of how a concept is formed theoretically.

Keywords

abstraction; reflective abstraction; APOS; Abstraction in Context

Introduction

Abstraction is a process in construction concepts, so that abstraction is the primary cognitive process in mathematics learning (Damerow 1996; Hershkowitz, Schwarz, and Dreyfus 2001; Scheiner and Pinto 2014a). Through abstraction, learners experience the transformation of knowledge on new concepts at a higher ((Damerow 1996; Gray 2007; Hershkowitz et al. 2001). Specifically, Piaget stated that cognitive structures developed through reflective abstraction solely support mathematical logic (Arnon et al. 2014a).

Several abstraction theories have been chronologically stated. For instance, Piaget proposed the empirical, pseudo-empirical, and reflective abstraction theories (Piaget 1977). Hershkowitz, Schwarz, and Dreyfus (2001) published a study entitled Abstraction in Context theory and the RBC+C model, and David Tall (2003) introduced structural abstraction. Scheiner criticized the David Tall's analysis of structural abstraction by stating that the theory only discusses the level of complexity, compared to abstraction, which investigates the construction of mathematical concepts (Scheiner and Pinto 2014b).

Meanwhile, Piaget's theory of abstraction was continued by Dubinsky by adjusting and adopting current conditions, with the support of the results of the continuous development of empirical studies through APOS theory (Arnon et al. 2014a). This development makes APOS a dynamic and context-rich theory with a strong identity. Conversely, the theory of Abstraction in Context offers a new concept that integrates the notion of past and present mathematics learning (Scheiner and Pinto 2014a). This theory has been developing from abstraction processes to pedagogical approaches and research methods.

This is a literature study, aims to comprehensively investigate the peculiarities of APOS and AiC as models for constructing mathematical concepts through abstraction. This study is not to determine the most effective model but to highlight the fundamental processes in mathematics learning that have been neglected, including how students construct mathematical concepts. This research focuses on (1) the fundamental theory of AiC and APOS epistemology, (2) the abstraction process used to propose mathematical concepts, (3) pedagogical aspects, and (4) appropriate research methods on the study of abstraction. The structure of this paper is that APOS theory will be present first than AiC, considering that APOS appears earlier than AiC.

Literature Review

Fundamental Theory of APOS and AiC Epistemology

There are several definitions of abstraction. Skemp (1971), for example, defines abstraction as an activity in which we are aware of the similarities between our experiences. Meanwhile, Locke (von Glaserfeld, 1991) states that abstraction takes ideas from a particular thing and becomes a general representation of the same kind, given a general name, and applies to anything that fits the abstract idea. In contrast, Piaget did not believe that abstract ideas are obtained by excluding object's general characteristics. Piaget was presumed that conceptual knowledge is obtained by performing particular object's actions and operations. In other words, knowledge of an object, both mentally and physically, is based on its subject or action.

APOS theory is based on Piaget's argument that emphasizes the fact that cognitive concepts and mathematical understandings are developed through reflective abstraction and theoretical problems solved in a social context to rebuild and organize specific mental structures in a schema (Lerman, 2020). In general, the process of reflective abstraction, according to Piaget, can be understood as follows: the learner observes the results of actions performed on objects, whatever produces logico-mathematical experiences. The results of this experience are interpreted by the scheme of action developed by the learner. In order to observe it, the learner performs another action, using the same scheme with which the interaction is to be considered. However, this form is new to the learner because the logico-mathematical experience teaches something not consciously realized before. Therefore, the abstraction by which the learner delivers new knowledge involves construction. This construction, or reconstruction, replaces experience or empirical procedures for the learner in a new plane.

Dubinsky et al. expanded APOS from the characteristics of Piaget's reflective abstraction theories: (1) focus on action and (2) action on mental objects (Scheiner & Pinto, 2014b). So that, the fundamental assumption of the APOS is based on the three basic types of mathematical knowledge: actions, processes, and objects. Dubinsky introduced the basic idea in the 90s, and after that, the idea evolved through much research. The term APOS was first introduced in 1997 by Cottrill. Meanwhile, Abstraction in Context (AiC) developed from the definition of abstraction as an activity that vertically reorganizes the previously owned mathematical structure into a new one (Dreyfus, 2014). The word "new"

means that the subject is conceptualizing something that was previously inaccessible. Conversely, the term activity emphasizes that abstraction is a historical process adopts other tools or equipment and occurs in specific social settings (Hershkowitz et al., 2001).

AiC theory comes from Bart van Oer's disagreement with the separation of context from abstraction (Hershkowitz et al., 2020). The research carried out by Van Oers is consistent with Davido's view, which stated that scientific knowledge is not an extension of student's daily experiences. It requires the cultivation of an idea and its resultant internal connections. According to Van Oers, abstraction is the process of instilling knowledge not from concrete to abstract, instead of from an undeveloped to a more developed, consistent, and elaborated form (Dreyfus et al., 2015). In this process, students need to adopt a socio-cultural approach and a design that bridges the different perspectives of the teacher and student. Rina Hershkowitz, Tommy Dreyfus, and Baruch B. Schwarz have been proposing Abstraction in Context since the 2000s. And now, where Abstraction in Context is known as a theoretical framework for studying students' processes of constructing abstract mathematical knowledge, which occurs in a context that includes specific curricular and social components and a learning environment (Dreyfus et al., 2015).

Abstraction Process as a Mathematics Conceptual Construction

Student's understanding of abstract mathematical knowledge is a significant concern in the study of mathematics. Under the APOS theory, the understanding is achieved through mental mechanisms that produce a structure consisting of Action (A), Process (P), Object (O), and

Schema (S). Action is described as a series of instructions explicitly carried out to change a mental or physical object. Process is a mental structure that performs similar operations as an action; it entirely takes place in the individual's mind. When individuals are aware of the overall process, Object develops. Schema is the final mental structure constructed when individuals coherently organize and link Actions, Processes, and Objects to a particular topic.

The types of mental mechanisms that lead to the APOS mental structure are:

1. Interiorization occurs when the individual repeats, reflects and combines action with another; it is interiorized into a mental structure.
2. Coordination is the composition of two or more Processes to build a new one,
3. Encapsulation occurs when an individual is aware of the entire Process and its transformation, which converts the initially dynamic procedure into a static object, forming its mental structure.
4. Reversal is a mechanism that creates a new Process by reversing the operation of an existing procedure.
5. Generalization is a mechanism that applies Processes in a more general context. However, when an individual learns to use schema to a broader phenomenon, it is assumed that the schema has been generalized. The Schema remains the same except that it now has wider applicability. Piaget called it reproductive, assimilation, or extensional generalization.
6. Thematisation is the last mechanism, and it occurs when individuals construct an object by applying actions and processes to the schema.

The mental mechanisms show that there are several ways/mechanisms to construct a mental structure. For example, the

Process is formed through interiorization, coordination, or reversal mechanisms. These mechanisms do need to be completely fulfilled to enable its formation through interiorization.

On the contrary, the Abstraction in Context (AiC) theory is developed from the notion of abstraction, which contains two keywords: vertical and the emergence of a new structure. Therefore, mathematical knowledge in AiC theory emerges through three stages, i.e., (1) the need for new constructions, (2) its emergence, and (3) consolidation (Tabach et al., 2018). The emergence of new constructions by the subject is explained and analyzed with the RBC+C model, comprising three observable mental epistemic actions: Recognizing, Building-with, and Construction as the core action (Hershkowitz et al., 2001; Tsamir & Dreyfus, 2002).

The Recognition action causes students able to identify that problems related to their knowledge were initially developed. In the Building-with stage, these constructs are combined to solve the problem. An individual assembles and integrates previous constructs through vertical mathematics to produce the new one in the Construction stage. The Construction is the final part of the abstraction process (Ozmantar & Monaghan, 2006), and the emergence of a construct is the outcome of the RBC process. However, this does not mean that it is bound to stick forever because this construct is often fragile. Although, when the use of the construction becomes more apparent, the Consolidation process is presumed to have occurred. This consolidation is evident in the student's progressive abilities, such as quickly recognizing the relevance of constructs and using them more flexibly in subsequent activities (Tsamir & Dreyfus,

2002). Therefore, the consolidation in question is not about abstraction; rather, it is based on constructs. Figure 1 below

shows the APOS and AiC abstraction processes.

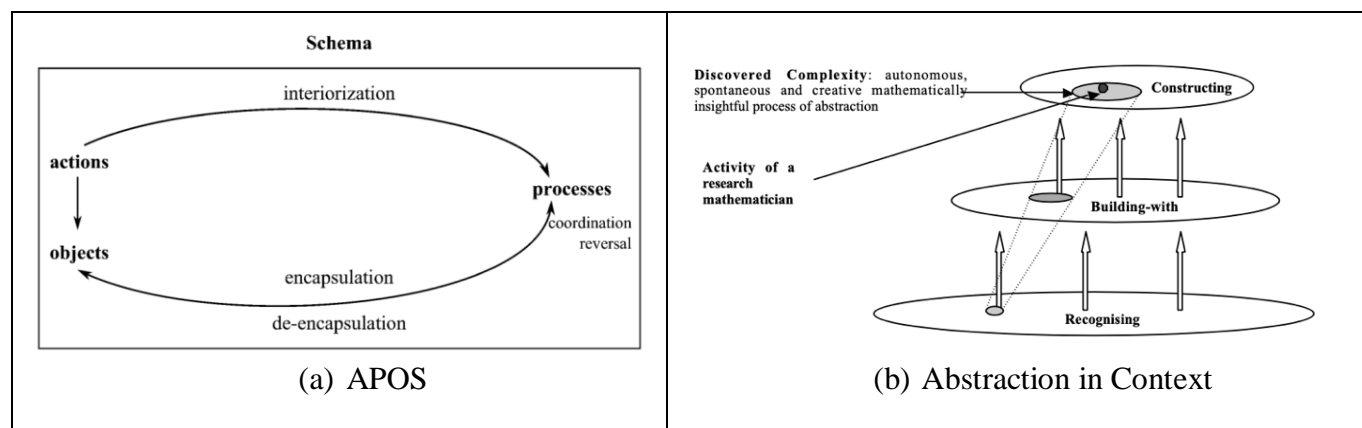


Figure 1. The AiC and APOS Abstraction Process

Pedagogical Approach to Foster Abstraction

Abstraction is not something that the teacher offers; rather, the students themselves develop it. To create reflective abstraction, students need to carry out a mental or physical activity concerning the subject. Therefore, the teacher plays a role in designing a learning process that creates opportunities for students to build their abstractions. Educators create situations and apply instructional methods that enable or encourage students (Dubinsky, 1991; von Glaserfeld, 1991). In the view of APOS theory, concepts are understood by building the necessary mental structures. However, this occurs through both traditional and non-traditional pedagogical strategies (Cetin & Dubinsky, 2017).

There are two main stages that educators need to pay attention to in terms of building learning situations that foster abstraction. The first stage is to identify the mental structures necessary to develop concepts and design appropriate learning

processes. The identification of mental structures is called genetic decomposition. The second stage is to build learning designs and instructions following the ACE pattern, a teaching cycle consisting of Activities, Classroom Discussion, and Exercise stages. According to the APOS theory, ACE is a pedagogical strategy that encourages students to build a mental construction.

Activity (A) is a the cooperative efforts of students working in teams to execute tasks designed to shape their mental construction, hypothesized through genetic decomposition. The focus is not only to get the correct answer. Instead, it also encourages the emergence of reflective abstractions according to the identification of the genetic decomposition. Therefore, Dubinsky recommends the use of a brief computer program for this stage. The second part is classroom discussion (C) which involves a small group of students, led by the teacher. In this session, they are allowed to reflect on their answers. The last is the exercise (E) stage, where

students work on problems individually to reinforce previous class activities and discussions. This stage helps to support the continued development of mental constructs exhibited by genetic decomposition.

In the Abstraction in Context paradigm, as the name implies, to foster abstraction, mathematical, curricular, historical, and social contexts are critical in RBC-based learning. The choice of social context and interaction varies significantly according to the teacher's decision. Furthermore, educators need to carry out preliminary analysis to convert the RBC model into a learning process by identifying the required knowledge.

In terms of learning strategies, Dreyfus (2015) suggested that it needs to be carried out in groups of 2 to 4 people to support the emergence of abstractions (Hershkowitz et al., 2007). This grouping is to reinforce the four principles related to growing abstraction reported by Dreyfus:

- a. giving rise to cognitive conflict,
- b. creating collaborative situations,
- c. proposing hypotheses and providing tools to test, and
- d. expressing a reflective argument.

Research Methods in Study of Abstraction

The APOS theory and the AiC model provide complete and detailed information regarding the study of procedures on abstraction. Several preliminary studies have also been implemented both at various levels, ranging from elementary (Arnon et al. 2014b; Celebioglu and Yazgan 2015; Sümen 2019) to junior high school (ÇubukluÖz et al. 2018; Hariharan 2016; Rahmawati, Budayasa, and Ekawati 2018) senior high school (Hassan and Mitchelmore n.d.; Mandasari, Arnawa, and

Atmazaki 2018), and undergraduate (Hanifah and Irsal 2019; Maharaj 2010; Santos 2019; Subroto and Suryadi 2018).

The research method for analyzing the APOS framework involved three components that influence each other: (1) theory analysis, (2) instruction design and implementation, and (3) data collection and analysis. The first component is theoretical analysis, where a study is carried out on the cognition of the mathematical concepts to be developed. The output of this step is preliminary genetic decomposition, a description of the mental mechanisms that people produce in constructing their understanding of mathematical concepts. The mental mechanisms built on the genetic decomposition are further translated into various activities or exercises that need to be carried out by students, as well as pedagogical strategies that have to be adopted by the educators to achieve mental mechanisms, such as cooperative learning, small group discussions, etc.

After determining the instruction design, the third component is data collection and analysis, which involves gathering empirical evidence for genetic decomposition resulting from the first phase. Without data collection, genetic decomposition is only a hypothesis. The data collection instruments include written questionnaires, semi-structured interviews, exams, computer games, class observations, textbook analysis, and studies on history or epistemology. Data analysis aimed at answering the following research questions (1) Have students developed mental constructions as described in genetic decomposition?, and (2) How well do students learn about concepts?. Assuming that the answer to the first question is no, then the instructions designed in the first step need to be reconsidered and revised. The cycle is repeated until these questions are positively answered, and it is confidently

confirmed that the students have learned the mathematical concepts. The data analysis is triangulated through collaborative research. Therefore, the results are negotiated using more than one instrument till their interpretation reaches a consensus.

The stages in the AiC show coherent and technically detailed steps to conduct the research of abstraction, which start with determining the abstraction design and compiling a priori analysis according to the concept to be built and end by testing the developed constructs. Consist of seven stages, the research was designed to make sure that all epistemic actions of RBC+C were passed. The seven stages are:

1. Determining the design of abstraction. This stage purposes to create a didactic hierarchy that aims to rearrange student's knowledge vertically. This design limits the student's and teacher's actions.
2. Performing a priori analysis. It is carried out by identifying the elements of knowledge intended by design, usually in concepts or strategies in the mathematical content/materials domain. The results are expected to answer the question, "what knowledge is helpful or necessary to handle and complete tasks that satisfy the teacher?".
3. Collecting data and preliminary analysis. Data are collected through individual/group interviews, student work documents, transcripts, and field notes. Conversely, the preliminary analysis is carried out by first reading the entire transcript and watching videos to overview the whole learning process. Second, by trimming the transcript to obtain relevant data and further dividing it into several parts, the final aspect is determined by cognition, content, or external factors that influence both. Changes in

cognition occur when there is an alteration in student orientation, for example, in the questions or methods adopted. Content changes happen when there is a new task transition. In addition, external factors are usually adopted by the teacher when controlling the class. The third preliminary analysis is to identify the emergence of new knowledge.

4. Identifying student's cognitive needs. This occurs through cognitive conflict, which is manifested through enthusiasm, uncertainty, surprise, curiosity, confusion, or a request for more time to reflect or absorb the situation at hand. Educators need to be observant to identify every phenomenon related to the need because, at this stage, students are not aware.
5. Accordant analysis to RBC model. This analysis aims to reveal how the students' new constructs emerge as a vertical reorganization of the previous one, and contribute to the AiC improvement through the unfolding of processes during this episode. The analysis of Recognition, Building with, and Construction epistemic action is based on the preliminary analysis compiled in the previous stage. It involves identifying and marking the relevant speech in the transcript, working backward through it to reorganize and develop contributory actions. Table 1 shows a table to determine the process of epistemic action that the subject achieve (Dreyfus, Hershkowitz, and Schwarz 2015).

Table 1. A hypothetical table of epistemic action

No	Name	Utterance	C ₁	C ₂
(1)	(2)	(3)	(4)	(5)
1	A	...		
2	B			
3	A			
4	B		R _a	
5	T			
6	B			
7	A			
8	B		B _a	
9	A		R _a B _a	
10	B			R _a
11	A		B _a	B _a
12	A			R ₁
13	B			B ₁
14	A			B ₁
15	B			B _a B ₁

Dreyfus explained that the first and second columns show the speech number and the speaking subject for the subject above. In the example above, there are three subjects: student A, student B, and the teacher (T). The third column contains the speech transcript of the subject. The fourth and fifth columns show the analysis of the first (C₁) and the second (C₂) construction actions. In column C₁, R_a and B_a indicate Recognizing and Building the C_a construction, a pre-existing construct. Columns C₂, R₁, and B₁ show epistemic action of Recognizing and Building for the first construct (C₁). This pattern indicates that the R_k and B_k cannot occur in the C_k column because the construct has not yet been formed in the students' minds. Rows in column C_i are filled in according to the contributions of the speech delivered by the subject (Dreyfus et al. 2015).

6. Consolidation

After the construct is formed due to the abstraction process, its existence is tested through consolidation. This is

necessary because students tend not to be aware of the construction process. However, through this, students' awareness of the existence of a new construct is tested, including whether they can use it to solve the following problem.

7. See the person who developed the Construction

It was advised not to conduct direct interviews with students to identify the individual responsible for the knowledge construction because it potentially leads to excessive intervention, thereby getting them trapped. Through group work and communicating with 3 to 4 people, identification is suggested to emerge an interactive flow among students, enabling them to develop further knowledge.

Methodology

This study used the literature method with the steps: (1) finding literature that matches the APOS and Abstract in Context themes, (2) selecting the appropriate literature, (3) analyzing the literature content, and (4) presenting the results of the analysis.

Results dan Discussion

Foundation Epistemology

APOS and AiC provide rigorous fundamental theories that have been proven and used in many types of research. Both of them rely on the assumption that concepts and mathematical knowledge are developed through abstraction. The difference is APOS comes from the concept of Piaget's reflective abstraction, while AiC comes from van Oer's abstraction as the process

of instilling knowledge from an undeveloped to a more developed one.

The Process of Abstraction

The similarity between APOS and RBC models from AiC theory is that concept construction runs linearly from Action – Process – Object – Schema and from R to B and then from B to C. Both are sequentially operated without skipping. However, theoretically, this linearity is a way of developing new concepts or knowledge. The RBC model states that students often do not realize when the processes occur within them. Dubinsky further noted that the linearity in APOS does not happen in all cases.

Another similarity is the aspect of awareness developed through knowledge. In AiC, awareness determines the strength of the construction, which is tested in the consolidation stage. Meanwhile, for APOS, awareness is realized in the formation of objects. According to Piaget, this embodiment of the reflective abstraction characteristics, consisting of two inseparable elements: *reflechissement* and *reflexion*. *Reflechissement* is a projection of something borrowed from a lower to a higher level, while *reflection* is an awareness of cognitive reconstruction or reorganization of the transferred knowledge. This abstraction reflection is observed at all stages using the sensory-motor (von Glaserfeld, 1991). The two figures mentioned above show that verticalization occurs in every step of AiC, from Recognizing to Building with and Construction. Conversely, in APOS, it appears after the schema has been constructed.

Another difference is that in APOS, the output of each abstraction process (mental structures) is clearly defined in terms of Process, Object, and Schema. This means

that the Action, Process, Object, and Schema structures are possible to exceed even though students do not experience all of the six mental mechanism processes. This is because the mental structure is achieved through several mechanisms. Figure 1(b) shows that the process structure tends to be achieved through encapsulation, coordination, or reservation mechanisms. However, in AiC, to embrace new knowledge, the subject needs to first go through all three RBC epistemic actions. Therefore, a schema is the final output of APOS, while in RBC + C, a construct is the final output of the abstraction process. 085834465832

Pedagogical Aspect to Support Abstraction

The descriptions of the pedagogical designs of APOS and AiC above show certain similarities between AiC and APOS, where both suggest the adoption of group work during the learning process. These theories also required identifying thought processes called genetic decomposition in APOS and preliminary analysis in AiC. These are the first steps that need to be adopted by educators because this identification serves as a guide for learning activities.

Research methods

Based on the designs of research of APOS and AiC, it is evident that the study of abstraction requires an initial framework relating to developing mathematical knowledge or concepts. In APOS theory, this framework is called genetic decomposition and in AiC theory it's called preliminary analysis. Both structures play a similar role, as the basis for determining pedagogical design and instruction. The subsequent learning stage is determined based on a priori decomposition or analysis compiled.

In data collection, APOS and AiC theory recommend a qualitative approach with triangulation analysis. AiC comprehensively explained data collection and analysis by advising the educators not to conduct direct interviews with students because it leads to excessive intervention; hence students were trapped. Furthermore, the AiC's design also shows the stages of more detailed analysis through in table 1.

Conclusion

1. Abstraction is a conceptual construction process that transformed students' knowledge to a higher level.
2. Dubinsky et al. developed APOS theory based on the reflective abstraction concept, which focuses on mental objects. Dreyfus et al. developed the AiC model based on the definition of abstraction as an activity that vertically organizes pre-existing mathematical structures into new ones. Both concepts emphasize action and activity, which shows that the construction of mathematical concepts emerges from the active participation of the students.
3. The concept construction or abstraction processes take place in stages. In APOS theory, mental mechanisms (interiorization, coordination, encapsulation, reversal, generalization, and thematization) construct mental structures Action, Process, Object, and Scheme. In AiC theory, the the construction stage is realized through epistemic actions, namely Recognizing, Building with, and Construction. An individual or subject will test the existence of his/her construct in the consolidation stage.
4. The abstraction process produces a new concept in APOS and AiC called

schema and construct. The emergence of this schema or construct is a form of knowledge verticalization.

5. Abstraction does not appear by itself. The teacher needs to create a design to realize a supportive learning situation, causing students to experience the construction process. To create the learning design, both APOS and AiC stated the need to identify the way the desired concept was formed theoretically. This identification process is called genetic decomposition in APOS and a priori analysis in AiC theory.
6. APOS and AiC suggested group learning that creates collaborative situations as a learning approach or strategy to foster abstraction. In particular, APOS recommends the ACE method, consist of Activity, Classroom Discussion, and Exercise stage.
7. To analyze abstraction, APOS and AiC provide detailed specific research designs. AiC complements this design with an instrument to identify the achievement of epistemic actions
8. APOS and AiC create avenues for further research relating to the way students construct mathematical concepts and provides broader opportunities for other studies to examine mathematical cognition through APOS and AiC on various contents.

Acknowledgement

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